# Weighted trade network in a model of preferential bipartite transactions

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Using a model of wealth distribution where traders are characterized by quenched random saving propensities and trade among themselves by bipartite transactions, we mimic the enhanced rates of trading of the rich by introducing the preferential selection rule using a pair of continuously tunable parameters. The bipartite trading defines a growing trade network of traders linked by their mutual trade relationships. With the preferential selection rule this network appears to be highly heterogeneous characterized by the scale-free nodal degree and the link weight distributions and presents signatures of nontrivial strength-degree correlations. With detailed numerical simulations and using finite-size scaling analysis we present evidence that the associated critical exponents are continuous functions of the tuning parameters. However the wealth distribution has been observed to follow the well-known Pareto law robustly for all positive values of the tuning parameters.

## I. INTRODUCTION

In a society different traders do business among themselves. Thus the wealth distribution of the society dynamically evolves through this trading process. In the simplest possible situation pairs of traders make economic transactions. Such a mutual interaction can be looked upon establishing a connection between them. Consequently one can define a trade network where each trader is a node and a link is established between a pair of nodes when the corresponding traders make a mutual trading. In this paper, we study the growth and the structural properties of a trade network in the framework of a well-known model of wealth distribution, namely, the kinetic exchange model (KEM) with quenched random saving propensities [1,2].

Over the last decade a lot of research efforts have been devoted to study the structures, properties and functions of different real-world as well as theoretically defined model networks [3]. The key characteristic features of these networks include their small-world properties, which simply implies the existence of a very short global connectivity even when the sizes of the networks are extremely large [4]. Second, there are a large class of networks that are extremely heterogeneous. Their heterogeneity are quantified by their degree (number of links meeting at a node) distributions. Usually such networks are observed to have power-law decay of their degree distributions and are called scale-free networks [5]. It has also been apparent that the links of many of these networks appear with a wide variation in strengths. In graph theoretic language the link strengths are called the "weights" in general. Such weighted networks have also been studied in the context of the passenger traffics of airport networks [6,7], international trade networks, etc [8,9].

More than a century ago Pareto proposed that the distribution of wealth $x$ in a society to be $P(x) \sim x^{-(\nu+1)}$ [10]. This form of distribution is generally known as Pareto distribution for a value of $\nu \sim 1$ [11]. Pareto suggested that $\nu = 1$ for the

wealth distribution and it is known as the Pareto law.

Application of the concepts of statistical physics to the wealth/income distribution in a society goes back to 1931 when Saha and Srivastava had suggested that the form of the wealth distribution may be similar to the Maxwell-Boltzmann distribution of molecular speeds in an ideal gas [12,13]. Over the last few years renewed attempts have been made using statistical physics methods. The main objective is to reproduce the recently collected individual income tax data in different countries reflecting the wealth distributions. It has been observed that these data fit well to exponentially decaying functions for small wealth which however ends with power-law tails in the large wealth regime.

Drăgulescu and Yakovenko (DY) [14] modeled a bipartite trading between two traders using the analogy of a pair of gas molecules interacting through an energy conserved elastic collision in an ideal gas. Starting from an arbitrary distribution of individual wealth the system evolves through a series of bipartite trades to arrive at a stationary state where the wealth distribution assumes its time independent form. While the DY model produces only an exponentially decaying wealth distribution, later modifications were suggested for its improvement. This class of models are now referred as the KEMs [1,2].

In a KEM the society is considered as a collection of $N$ traders, each having a certain amount of money equivalent to his wealth $x_i, \{i=1,N\}$ which he uses for mutual trades with other traders. Generally all traders initially start with an equal amount of money $P(x_i, t=0) = \delta(x_i - a)$. The sequential time $t$ is the number of bipartite trades. A trade consists of two parts, (i) a rule for the selection of a distinct pair of traders $i$ and $j$, $(i \neq j)$ and (ii) a distribution rule for sharing their total money $x_i + x_j$ between them. Different KEMs differ from one another either in the selection rule or in the distribution rule or in both. In the DY model a pair of traders is selected with uniform probability. Their total money is then randomly reshuffled between them. In the stationary state the wealth distribution is $P(x) = \exp(-x/\langle x \rangle)/\langle x \rangle$ where the mean wealth $\langle x \rangle = a$ is usually set at unity [14]. Quite naturally a trader invests $a$ only a part of his money in a trade and not his entire wealth. To incorporate this fact Chakraborti and

Chakrabarti (CC) introduced a saving propensity λ [15] same for all traders. As a result the stationary state wealth distribution gets modified to a distribution with a single maximum which approximately fits to a Gamma distribution [16].

In a second modification of the DY model Chatterjee, Chakrabarti, and Manna (CCM) assigned a quenched distribution of saving propensities $\{\lambda_i\}$ so that each trader is characterized by his own $\lambda_i$ value [17]. Using the same selection rule as in DY model, the total money invested by a pair of traders after saving has been randomly shared between them. The system reaches a stationary state here as well but it sensitively depends on the precise values of $\lambda_i$s. The wealth distribution in the stationary state after averaging over different realizations of the quenched disorder $\{\lambda_i\}$ yields a power-law decay with a value of the Pareto exponent $\nu \approx 1$ [17]. However, subsequent detailed analysis have revealed that the CCM model has many interesting features [18]. For example, in contrast to the DY and CC models, CCM model is not ergodic. Therefore the wealth distribution is not self-averaging and the single trader wealth distribution is totally different from the over all wealth distribution of the whole society. Consequently the individual saving propensity factor $\lambda_i$ plays the role of an identification label that determines the economic strength of a member in the society [18]. In fact, the wealth of a trader fluctuates around a mean value which depends very sensitively on the precise value of $\lambda_i$. Larger the value of λ higher is the mean wealth. Truly the wealth distribution averaged over many uncorrelated quenched $\{\lambda_i\}$ sets is the convolution of the individual members' wealth distributions [18]. This overall distribution for the whole system exhibits Pareto law but not the individual member distributions. The exponent $\nu = 1$ has been found to be exactly equal to unity in [19,20].

In Sec. II we describe our modification of the CCM model using the preferential selection rule. The wealth distribution of the modified CCM model has been described in Sec. III. In Sec. IV we define the associated trade network in terms of its nodal degrees and link weights. Section V presents the degree distribution and the weight distribution is discussed in Sec. VI. We summarize in Sec. VII

## II. MODEL

It is a general observation that in a society the rich traders invest much more in trade and therefore take part in the trading process more frequently than the poor ones. To incorporate this fact in the CCM model that rich traders are preferentially selected more frequently with higher probabilities we introduce two parameters $\alpha$ and $\beta$ in general, both $\geq 0$, to tune the preference different traders receive for their selections. We assume that the probability of selection of a trader is directly proportional to the $\alpha$ and $\beta$-th power of its wealth. Therefore a pair of traders $i$ and $j$ ($i \neq j$) with money $x_i$ and $x_j$ are selected with probabilities

$$\pi_i(t) \propto x_i(t)^\alpha \quad \text{and} \quad \pi_j(t) \propto x_j(t)^\beta. \tag{1}$$

When $\alpha = \beta = 0$ we get back the CCM. When they are nonzero the rich traders are selected with larger probabilities. More rich a trader, higher is the probability that it will be

selected for trading. Once a pair of traders $i$ and $j$ is selected, they save $\lambda_i$ and $\lambda_j$ fractions of their money and invest the rest amounts to the mutual trade. The total invested amount by both the traders is therefore

$$\delta_{ij}(t) = \bar{\lambda}_i x_i(t) + \bar{\lambda}_j x_j(t), \tag{2}$$

where $\bar{\lambda} = 1 - \lambda$. This amount is then randomly divided into two parts and received by them randomly,

$$x_i(t+1) = \lambda_i x_i(t) + \epsilon(t)\delta_{ij}(t),$$

$$x_j(t+1) = \lambda_j x_j(t) + \bar{\epsilon}(t)\delta_{ij}(t), \tag{3}$$

where $\epsilon(t)$ is a freshly generated random fraction and $\bar{\epsilon} = 1 - \epsilon$.

It is essential that all measurements are done once the system relaxes to the stationary state. For that it is necessary that a number of transactions take place between every pair of traders, only then the mean wealth of traders attain their stationary values and fluctuate around them. Equation (1) states that for any $(\alpha, \beta) > 0$ the richest and the next rich are the most probable pair and the poorest and the next poor are the least probable pair for selection. Assuming the maximum wealth $x_{max} \sim N$ (with $\langle x \rangle = 1$) and the minimal wealth $x_{min} \sim 1/N$ the relaxation time can be estimated which is the typical time required for the poorest pair to make a trade. The poorest is selected with a probability $x_{min}^\alpha / \Sigma_i x_i^\alpha$. Approximating the denominator by its maximum value we get $(x_{min}/x_{max})^\alpha \sim N^{-2\alpha}$. Similarly the probability for selecting the next poorest is $N^{-2\beta}$ and for the poorest pair is $N^{-2(\alpha+\beta)}$. Therefore, the time required for a trade between the poorest pair $T_2 \sim N^{2(\alpha+\beta)}$ (see below) and the relaxation time is several multiples of $T_2$. Thus for any $(\alpha, \beta) > 0$ the relaxation time grows very rapidly with $N$.

At the early stage rich traders at the top level quickly take part in the trading but gradually the inclusion of relatively poor traders becomes increasingly slower. As a result the number of distinct traders taking part in the trading process grows very slowly. Effectively this implies that the system passes through a very slow transient phase which is practically time independent. We call this state as the "quasistationary state (QSS)." It is to be noted that in the following sections we present our numerical results for large system sizes in the QSS only. To ensure that the system has indeed reached the QSS we keep track of the sum $\Sigma_i x_i^2$ in our simulations and collect the data only after no appreciable change in its mean value is noticed. We mostly analyze the symmetric $\alpha = \beta$ cases except for a few asymmetric cases.

## III. WEALTH DISTRIBUTION

For CCM it was observed that the average money of a trader $\langle x(\lambda) \rangle$ with saving propensity λ diverges as $\lambda \to 1$ [18]. Later it was shown that the divergence is like $\langle x_i \rangle (1 - \lambda_i) = \text{const}$ [19,21]. This is likely since had there been a trader with $\lambda = 1$, he would not invest any money at all but always receives a share of the investments of the other traders. As a result this trader will eventually grab all the money of the entire society and this situation is similar to the phenomenon of condensation.
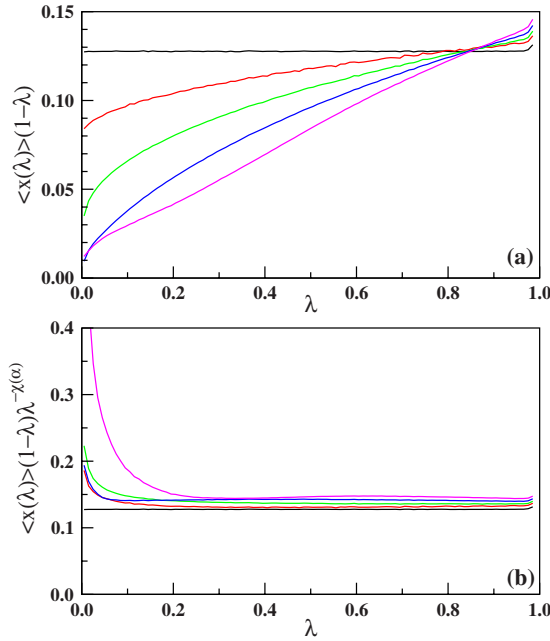
FIG. 1. (Color online) (a) Plots of $\langle x(\lambda)\rangle(1-\lambda)$ vs $\lambda$ for $\alpha=\beta=0$ (black), 1/2 (red), 1 (green), 3/2 (blue), and 2 (magenta) for $N=1024$ ($\alpha$ increases from top to bottom). (b) The product function $\langle x(\lambda)\rangle(1-\lambda)\lambda^{-\chi(\alpha)}$ is plotted with $\lambda$ using $\chi(\alpha)=0.15$, 0.35, 0.57, and 0.80 for $\alpha=1/2$, 1, 3/2, and 2, respectively, using the same colors as in (a).

In Fig. 1(a) we plot the quantity $\langle x(\lambda)\rangle(1-\lambda)$ vs $\lambda$ for five different values of $\alpha=\beta=0$, 1/2, 1, 3/2, and 2. For $\alpha=\beta=0$ we see the horizontal line as observed in [21]. However for other $\alpha,\beta$ values the variations in the same quantity are far from being uniform and are monotonically increasing with $\lambda$, their growth becoming increasingly faster with $\alpha$. Therefore we try multiplying this function by $\lambda^{-\chi(\alpha)}$ where $\chi(\alpha)$ is a function of the parameter $\alpha$. In Fig. 1(b) we plot $\langle x(\lambda)\rangle(1-\lambda)\lambda^{-\chi(\alpha)}$ vs $\lambda$ using the same data of Fig. 1(a) with $\chi(\alpha)=0.15$, 0.35, 0.57, and 0.80 for $\alpha=\beta=1/2$, 1, 3/2, and 2, respectively. We get nearly uniform variations between $\lambda=0.3$ and 1. We assume that

$$\langle x(\lambda)\rangle(1-\lambda)\lambda^{-\chi(\alpha)} = \text{const}. \tag{4}$$

If the distribution of $\lambda$ values is denoted by $g(\lambda)=$const, and since the wealth $x$ and the saving propensity $\lambda$ are the two mutually dependent variables associated with the same trader, their probability distributions follow the relation [19]

$$P(x)dx = g(\lambda)d\lambda. \tag{5}$$

Differentiating Eq. (4) with $\lambda$ one can find out the derivative $d\lambda/dx$ and substituting in Eq. (5) one gets

$$P(x) = \frac{C}{x^2}[\lambda^{-\chi} + (1-\lambda)\chi\lambda^{-\chi-1}]^{-1}. \tag{6}$$

For this equation we see that for large $\lambda$ the term within $[\cdots]$ is of the order of unity. Therefore in this range $P(x)\sim x^{-2}$ as in the Pareto law. This is an indication that even for $(\alpha,\beta)>0$, Pareto law holds good and in the following we present numerical evidence in support of that.
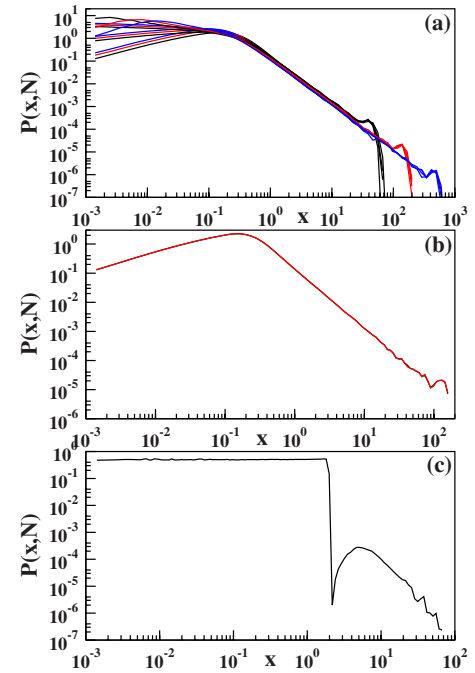


FIG. 2. (Color online) (a) Wealth distribution $P(x,N)$ vs $x$ for $\alpha=\beta=0$, 1/2, 1, and 2 and for $N=256$ (black), 1024 (red), and 4096 (blue) ($N$ increases from left to right). The slopes of these curves yield $\nu=1.00(3)$ consistent with the Pareto law. (b) $P(x,N)$ for $(\alpha,\beta)=(\infty,0)$ (black) and $(0,\infty)$ (red) for $N=1024$ which almost overlapped. (c) $P(x,N)$ for $(\alpha,\beta)=(\infty,\infty)$, the distribution is uniform followed by a hump due to transactions between the richest and the next richest traders only.

The system is prepared by assigning uniformly distributed random fractions for the saving propensities $\lambda_i$ to all $N$ traders. Here $\lambda_i$s are quenched variables and therefore they remain fixed during the subsequent time evolution of the trading system. Consequently all observable that we measured are averaged over different uncorrelated sets of the $\{\lambda_i\}$ values. While assigning the $\lambda$ values we first draw $N$ uniformly distributed random fractions, but then to avoid the situation when $\lambda_{\max}$ is very close to unity by chance we scale them proportionately so that $\lambda_{\max}=1-1/N$ in every $\{\lambda_i\}$ set. First a pair of values for $(\alpha,\beta)$ is selected. Two types of initial wealth distributions are used: (i) $x_i=1$ for all $i$ and (ii) $x_i$'s are uniformly distributed random numbers with $\langle x\rangle=$const. The sequence of bipartite trading begins by randomly selecting pairs of traders using Eq. (1). Once a pair is selected, their total individual invested amount $\delta_{ij}$ is calculated using Eq. (2) and this amount is shared again between them using Eq. (3). This constitute a single bipartite trading and the dynamics is followed over a large number of such trading events.

The wealth distribution changes with time from the initial distribution to more and more flat distribution. After a certain time the system passes through the quasistationary state when no appreciable change in the wealth distribution is observed. It is also observed that the distribution is robust with respect to the precise values of the parameters $\alpha$ and $\beta$ used. In Fig. 2(a) the wealth distribution $P(x,N)$ has been plotted with $x$ for four sets of parameter values namely, $\alpha=\beta=0$,

1/2, 1, and 3/2 and for three system sizes $N=256$, 1024, and 4096. Apart from slight fluctuations the four curves for a given system size nearly overlap on one another. On a double-logarithmic scale the slopes of the curves give an average estimate for $\nu=1.00(3)$ consistent with the Pareto law as observed in the CCM [17]. This indicates that the wealth distribution is robust with respect to the parameter values in this region. The nonzero values of $\alpha$ and $\beta$ only controls the frequencies with which different traders are called for trading.

Next we consider the case when one of the two parameters $(\alpha, \beta)$ is infinity and the other one is zero. If $\alpha=\infty$ the richest trader is always selected as the first trader. The other trader is selected among the other $N-1$ traders with uniform probability. As shown in Fig. 2(b) here also we see that the Pareto law holds good. For $\alpha=\infty$ and for finite $\beta$ first the richest trader is selected and then the second trader is selected with probability $\propto x_j^\beta$. We observe numerically that here also Pareto law works very well.

However the situation is very different when both $(\alpha, \beta)$ take very large values. In this situation almost always only the rich traders are called for transactions. The system passes through an extremely long QSS and the number of traders taking part in trade does not increase at all. For example in the limiting case of $(\alpha, \beta)=(\infty, \infty)$ it implies that always only the richest and the next richest traders are selected for transactions with probability one if their wealth are high enough and therefore the trading process does not involve any other trader. Therefore the wealth distribution for the single $\lambda_i$ set has two very high peaks and wealth of all other traders are small and uniformly distributed. Consequently the quench averaged wealth distribution is uniform throughout followed by a hump at the highest value of wealth [Fig. 2(c)]. A systematic analysis with many different $(\alpha, \beta)$ pairs leads us to conclude that Pareto law holds good in the positive quadrant of the entire $(\alpha, \beta)$ plane.

In Fig. 3 we exhibit this behavior in the positive quadrant of the $(\alpha, \beta)$ plane where Pareto law is valid and the limiting points are marked by circles with their characteristics. The origin at $(\alpha=0, \beta=0)$ represents the CCM model where traders are selected randomly with uniform probabilities. As explained below, the trade network corresponding to this point is a random graph. As explained before that at the two corners $(\infty,0)$ and $(0,\infty)$ the richest trader always participates in every transaction. Therefore the corresponding trade networks have starlike structures. In the last corner of $(\infty,\infty)$ the trade takes place only between the richest and the next rich traders and therefore the graph reduced to a single dimer only.

## IV. TRADE NETWORK

One can associate a network with this trading system. Each trader is a node of the network. Initially the network has only $N$ nodes but no links. First the system is allowed to reach the QSS and then the network starts growing. Every time a pair of traders makes a trade for the first time, a link is introduced between their nodes. There after no further link is added between them irrespective of their subsequent trades
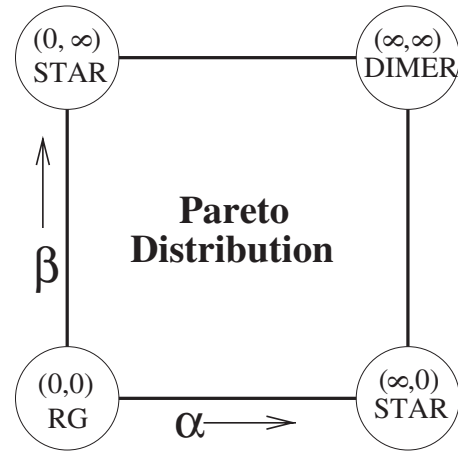


FIG. 3. The phase diagram in the positive quadrant of the $(\alpha, \beta)$ plane. The Pareto law is valid in the entire region. The origin corresponds to the CCM model where the trade network is a random graph. At the corners $(\infty,0)$ and $(0,\infty)$ the richest trader trade in every transaction, so that the network is a starlike graph but the wealth distribution still follows Pareto law as shown in Fig. 2(b). However at the corner $(\infty,\infty)$ the trade takes place between the top two richest traders and the network shrinks to a single dimer. The wealth distribution here is uniform followed by a hump as shown in Fig. 2(c).

and they remain connected with a single link. As the system evolves more and more traders take part in the trading dynamics and consequently the number of links grow in the network. For $\alpha=\beta=0$ the growth of the network is exactly the same as that of the random graph, however it is much different when $(\alpha, \beta)>0$. Since the rich nodes are preferentially selected they acquire links at a faster rate than the poor nodes. The degree $k_i$ of the node $i$ is the number of distinct traders with whom the $i$-th trader has ever traded. The dynamics is continued for a certain time $T$ till the average degree $\langle k \rangle$ of a node reaches a specific preassigned value.

In general there are two characteristic time scales involved in this problem. At the early stage the network grows with multiple components having different sizes. At time $T_1$ the growing network becomes a single component connected graph. A second time scale is $T_2$ when the whole network becomes a $N$-clique graph in which each node is linked to all others, which implies that each trader has traded at least once with all others. Unlike random graphs the growth of the network is highly heterogeneous and the rich traders have much larger degrees than the poor traders. Since poor traders are selected with low probabilities they take much longer times to be a part of the network. Consequently $T_1$ is found to be very large and of the same order as $T_2$. Numerically it is easier to calculate $T_2$, one keeps track of the number of distinct links and stops only when this number becomes just equal to $[N(N-1)]/2$. On the other hand to calculate $T_1$ one follows the growth of the giant component and stops when the giant component covers all $N$ nodes. A Hoshen-Kopelman cluster counting algorithm [22] is used to estimate the size of the giant component. For the ordinary CCM with $\alpha=\beta=0$ since both traders are chosen with uniform probability, the generated graph is a simple Erdős-Rényi random
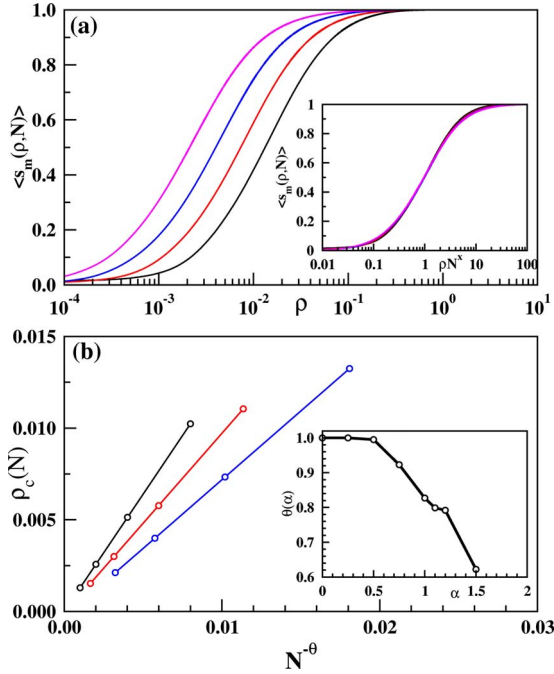
FIG. 4. (Color online) Growth of the trade network. (a) Plot of the average size of the giant component $\langle s_m(\rho,N) \rangle$ with the link density $\rho$, for $\alpha = \beta = 1$ and for $N = 128$ (black), 256 (red), 512 (blue), and 1024 (pink), ($N$ increases from right to left). The inset shows a data collapse of the same plots with $\rho N^\theta$, and $\theta = 0.88$. (b) The percolation link density $\rho_c(N)$ is plotted with $N^{-\theta(\alpha)}$ where $\theta(\alpha) = 0.88$, 0.92 and 1 for $\alpha = \beta = 1$ (black), 3/4 (red), and 1/2 (blue) respectively ($\alpha$ increases from left to right). The inset shows plot of $\theta(\alpha)$ with $\alpha$.

graph characterized by a Poissonian degree distribution [23].

The growth of the giant component is studied with increasing number of links $n$ in the in the trade network. The average fraction of nodes in the giant component is denoted by $\langle s_m(\rho,N) \rangle$ which is the order parameter in this percolation problem. This has been plotted in Fig. 4(a) using a semilog scale with link density $\rho = n/[N(N-1)]/2$ in the network. Four curves shown in this figure correspond to $N = 128$, 256, 512, and 1024 for $\alpha = \beta = 1$, the system size increasing from right to left. The inset shows that a data collapse can be obtained by scaling the $\rho$ axis by a factor $N^\theta$ with $\theta = 0.88$. The critical density of percolation transition $\rho_c(N)$ is defined as that particular value of $\rho$ for which the average size of the giant component $\langle s_m(\rho,N) \rangle = 1/2$. In Fig. 4(b) we show that how the critical percolation threshold $\rho_c(N)$ depends on $N$ by plotting it with $N^{-\theta}$ for $\alpha = \beta = 1/2$, 3/4, and 1. It has been observed that the exponent $\theta(\alpha)$ is dependent on $\alpha$ in general and in the inset of this figure we plot $\theta(\alpha)$ vs $\alpha$. We see that for $\alpha \leq 1/2$, $\theta(\alpha) = 1$ but for $\alpha > 1/2$, $\theta(\alpha)$ gradually decreases. For Erdős-Rényi random graph it is known that $\theta = 1$ and therefore this result gives an indication that the trade network is likely to be different from random graphs for $\alpha = \beta > 1/2$.

## V. DEGREE DISTRIBUTION

The degree distribution has been studied similar to random graphs. We keep track of the average degree $\langle k \rangle$ of the
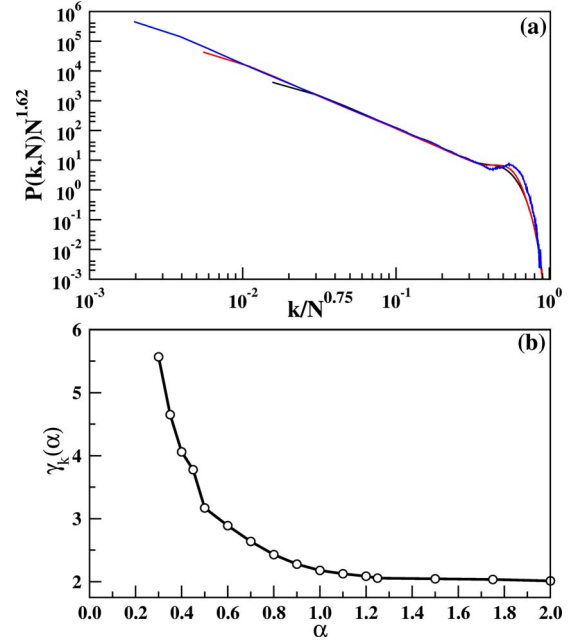


FIG. 5. (Color online) (a) The finite-size scaling of the degree distributions $P(k,N)$ with $\alpha = \beta = 1$ for $N = 256$ (black), 1024 (red), and 4096 (blue) and for $\langle k \rangle = 1$ ($N$ decreases from left to right). Direct measurement of slopes give $\gamma_k(1) = 2.18(3)$. The best data collapse corresponds to $\eta_k(1) = 1.62$ and $\zeta_k(1) = 0.75$ giving $\gamma_k(1) = 2.16(3)$. (b) The plot of $\gamma_k(\alpha)$ vs $\alpha$.

network which is related to the number of links $n$ of the network as $n = \langle k \rangle \frac{N}{2}$. First the degree distribution has been studied for $\langle k \rangle = 1$ and for different system sizes. For an assigned set of values of $(\alpha, \beta)$, for a given set of values for the saving propensities $\{\lambda_i\}$ and for a specific value of $N$ once $\langle k \rangle = 1$ is reached we calculate the degree distribution considering all components of the network on the same footing. The network is then refreshed by removing all links and a second network is constructed and so on. The dynamics is continued for the same values of the parameters and the same set of $\{\lambda_i\}$s till a large number of networks are generated and their mean degree distribution is calculated. The entire dynamical process is then repeated with another uncorrelated set of $\{\lambda_i\}$s and the degree distribution has been averaged over many such sets.

In Fig. 5(a) we show the finite-size scaling plot of the average degree distribution $P(k,N)$ vs $k$ for $\alpha = \beta = 1$ and for $N = 256$, 1024, and 4096. On a double-logarithmic scale all three curves show quite long scaling regions followed by humps before the cut-off sizes of the degree distributions. The cut-off sizes of the distributions shifts to the larger values of $k$ approximately by equal amounts on the double-log scale when the system size has been enhanced by the same factor. From the direct measurement of slope in the scaling region we estimate $\gamma(1) = 2.18(3)$. Almost the entire degree distribution obeys nicely the usual finite-size scaling analysis and an excellent collapse of the data is observed confirming the validity of the following scaling form:

$$P(k,N) \propto N^{-\eta(\alpha)} \mathcal{G}[k/N^{\zeta(\alpha)}], \qquad (7)$$

where the scaling function $\mathcal{G}(y)$ has its usual forms like, $\mathcal{G}(y) \sim y^{-\gamma(\alpha)}$ as $y \to 0$ and $\mathcal{G}(y)$ approaches zero very fast for

$y \gg 1$. This is satisfied only when $\gamma(\alpha) = \zeta(\alpha)/\eta(\alpha)$. The exponents $\eta(\alpha)$ and $\zeta(\alpha)$ fully characterize the scaling of $P(k, N)$ in this case. To check the validity of the equation we attempted a data collapse by plotting $P(k, N)N^{\eta(1)}$ vs $k/N^{\zeta(1)}$ by tuning the values of $\eta(1)$ and $\zeta(1)$. The values obtained for best data collapse are $\eta_k(1) = 1.62$ and $\zeta_k(1) = 0.75$ implying that in the infinite size limit $P(k, \infty) \sim k^{-\gamma(1)}$ with $\gamma(1) = 2.16(3)$. Tuning $\alpha$ and $\beta$ to other values it is observed that the degree distribution exponent $\gamma$ does depend on these two parameters. In Fig. 5(b) we show a plot of $\gamma_k(\alpha)$ with $\alpha$ which decreases to $\approx 2$ at $\alpha = 2$.

## VI. WEIGHTED NETWORK

Within a certain time $T$ a large number of bipartite trades take place between any arbitrary pair of traders. The total sum of the amounts $\delta_{ij}$ invested in all trades between the traders $i$ and $j$ in time $T$ is defined as the total volume of trade $w_{ij} = \Sigma^T \delta_{ij}$. Therefore $w_{ij}$ is regarded as the weight of the link $(ij)$. The magnitudes of weights associated with the links of the trade network are again found to be highly heterogeneous. This is primarily because within a certain time $T$ a rich pair of traders trade many more times than a rich-poor or a poor-poor pair. In addition the invested amounts depend on the mean wealth $\langle x_i \rangle$ of the traders involved as well as their saving propensity factors $\lambda_i$. The probability distribution $P(w, N)$ of the link weights are calculated when the average degree $\langle k \rangle$ reaches a specific preassigned value. As before, this distribution has also been averaged over many weighted networks for one $\{\lambda_i\}$ set and then further averaged over many uncorrelated $\{\lambda_i\}$ sets.

First we studied the case when the trade networks is a $N$-clique graph, i.e., when each trader has traded with all other traders at least once. Here each node has same degree, i.e., $P(k) = \delta[k - (N-1)]$ and $\langle k \rangle = N-1$. The required time $T_2$ increases rapidly with $N$ as described in Sec. II and we could study small system size $N = 64$ only. The distribution has a very long tail and therefore we used a lin-log scale for plotting. In Fig. 6(a) we show the plots of $P(w, \alpha)$ with $\ln(w)$ for different values of $\alpha = 0$, 1/4, 1/2, 3/4, and 1 and $\beta = 1$. Each curve is asymmetric and has a single maximum. The position of the peak shifts toward larger values of $\ln(w)$ as $\alpha$ increases. If Fig. 6(b) a similar plot has been shown for $\langle k \rangle = 5$ for three network sizes $N = 128$, 256, and 512 and for $\alpha = \beta = 1$. On a double-logarithmic scale each curve has a considerable straight portion. This indicates a power-law decay like $P(w, N) \propto w^{-\gamma_w}$. The corresponding slopes give estimates for the exponent $\gamma_w$ as 2.52, 2.53, and 2.51 for the three system sizes respectively, so that on the average $\gamma_w = 2.52(3)$.

The strength of a node $s_i = \Sigma_j w_{ij}$ where $j$ runs over all neighbors $k_i$ of $i$, is a measure of the total volume of trade handled by the $i$-th node. Nodal strengths varies over different nodes over a wide range. We first study the probability distribution of nodal strengths. In Fig. 7(a) the strength distribution $P(s, N)$ has been plotted for the average degree $\langle k \rangle = 5$ for $\alpha = \beta = 1$ and for the network sizes 256, 1024, and 4096. Extended scaling regions at the intermediate regions of the curves indicate that $P(s, N)$ also follows a power-law
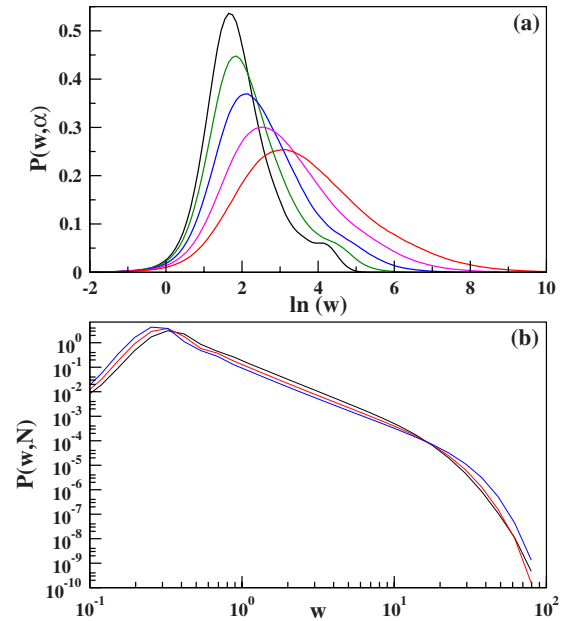


FIG. 6. (Color online) Probability distribution $P(w, \alpha)$ of the link weights. (a) Plots for the $N$-clique graphs with $N = 64$ and for $\alpha = 0$ (black), 1/4 (green), 1/2 (blue), 3/4 (magenta), and 1 (red) and $\beta = 1$ always (the maximum value decreases with increasing $\alpha$). (b) Plots for $\langle k \rangle = 5$ with $\alpha = \beta = 1$ and for the system sizes 128 (black), 256 (red), and 512 (blue) ($N$ decreases from left to right). Direct measurement of slopes gives 2.52, 2.53, and 2.51, respectively.

decay function $P(s, N) \sim s^{-\gamma_s}$ in the limit of $N \to \infty$. Direct measurements gives an estimate of $\gamma_s(1) \approx 2.5$. In Fig. 7(b) we try a similar finite-size scaling of the same data giving $\eta_s(1) = 1.64$ and $\zeta_s(1) = 0.67$ giving $\gamma_s(1) = 2.45(5)$.

Quite often weighted networks have nontrivial strength-degree correlations. Example of such networks are the airport networks and the international trade network. For a network where the link weights are randomly distributed, the $\langle s(k) \rangle$ grows linearly with $k$. However a nonlinear growth like $\langle s(k) \rangle \sim k^\phi$ with $\phi > 1$ exhibits the presence of nontrivial correlations. In Fig. 8(a) we plot the variation in $\langle s(k) \rangle$ vs. $k$ for a system size $N = 16 384$ and for different values of $\alpha = \beta$. The slopes of these plots give estimates for the exponent $\phi(\alpha)$ which gradually increased with $\alpha$ and this variation has been plotted in the inset. In the same context we also studied how the mean wealth of a trader depends on its degree. The mean wealth of a trader $\langle x(k) \rangle$ has been plotted in Fig. 8(b) with its degree $k$ for the same system sizes as in Fig. 8(a) and for the same values of parameters. A power-law growth has been observed for all values of $\alpha$: $\langle x(k) \rangle \sim k^{\mu(\alpha)}$. The slopes of these plots give estimates for the exponent $\mu(\alpha)$ which has been plotted in the inset of Fig. 8(b).

## VII. SUMMARY

To summarize, we have studied different structural properties of a trade network associated with the dynamical evolution of a model of wealth distribution with quenched saving propensities. In this model distinguishable traders make preferential bipartite trades among themselves and thus cre-
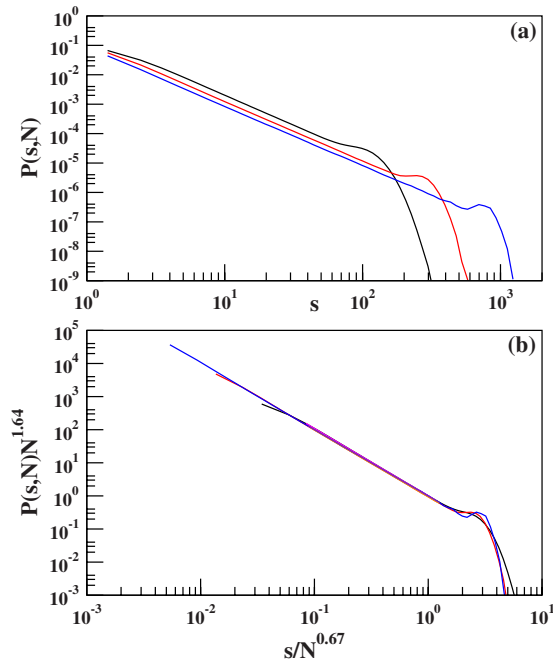
FIG. 7. (Color online) (a) Nodal strength distributions $P(s,N)$ with strength $s$ with $\alpha=\beta=1$, for $N=256$ (black), 1024 (red), and 4096 (blue) ($N$ increases from left to right) and for $\langle k \rangle=5$. Direct measurement of the slopes of these curves gives $\gamma_s=2.50(5)$. (b) The Finite-size scaling analysis of the data (using same colors) in (a) gives $\eta_s=1.64$ and $\zeta_s=0.67$ estimating the value of $\gamma_s=\eta_s/\zeta_s=2.45(5)$.



FIG. 8. (Color online) (a) Plot of nodal strength $\langle s(k) \rangle$ with degree $k$ for $N=16384$ and for $\alpha=\beta=1/2$ (black), 3/4 (green), 1 (blue), 5/4 (magenta), and 3/2 (red) ($\alpha$ decreases from left to right). The slopes $\phi(\alpha)$ of these curves are plotted in the inset. (b) Plot of average wealth $\langle x(k) \rangle$ with degree $k$ for $N=16384$ and for $\alpha=\beta=1/2$ (black), 3/4 (green), 1 (blue), 5/4 (magenta), and 3/2 (red) ($\alpha$ increases from left to right). The slopes $\mu(\alpha)$ of these curves are plotted in the inset.

ate links among themselves. They are selected for trade preferentially using a pair of continuously tunable parameters, where the rich traders are picked up more frequently for trade than poor traders. This creates strong heterogeneity in the system which has been reflected in the power-law distributions of the nodal degree and the link weight distributions measuring volumes of mutual trade. We present numerical evidence that the associ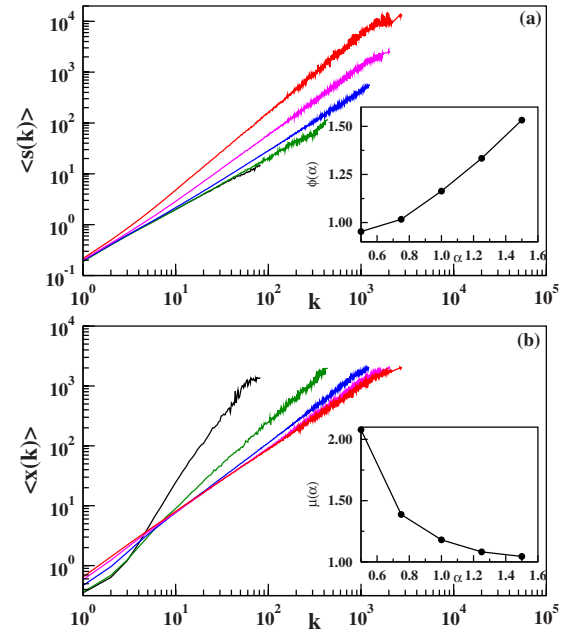ated individual wealth distribution follows the well-known Pareto law robustly for all positive values of the parameters.

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